Nirma University
Institute of Technology
Department of Mathematics & Humanities
B. Tech. (ALL) – Semester - I
Calculus and Differential Equations (MA102)
Questions for practice

Calculus (Gamma & Beta Function)

1) $\int_0^{\frac{\pi}{2}} cos^6 x\ sin^7 x\ dx$
2) Show that $\int_0^{\frac{\pi}{2}} sin^P\theta\ cos^q\theta\ d\theta = \frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{P+q+2}{2}\right)}$
3) Evaluate $\int_0^1 \frac{dx}{(1-x^n)^{\frac{1}{n}}}$
4) Evaluate $\int_0^\infty x^m e^{-ax^n} dx = \frac{1}{n}\frac{\Gamma\frac{m+1}{n}}{a^{\frac{m+1}{n}}}$
5) Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta \times \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{tan\,\theta}} d\theta$
6) Evaluate $\int_0^1 (x\log x)^4 dx = \frac{4!}{5^5}$
7) Show that $\int_0^1 x^m \left(\log\frac{1}{x}\right)^n dx = \frac{\Gamma(n+1)}{(m+1)^{n+1}}$
8) Show that $\int_0^\infty \sqrt{x} e^{-\sqrt[3]{x}} dx = \frac{315}{16}\sqrt{\pi}$
9) Show that $\int_0^1 x^m (\log x)^n dx = \frac{(-1)^n \Gamma(n+1)}{(m+1)^{n+1}}$
10) Show that $\int_{-\infty}^{\infty} e^{-h^2x^2} dx = \frac{\sqrt{\pi}}{h}$
11) Show that

$$\int_0^1 \frac{dx}{\sqrt{x\log\frac{1}{x}}} = \sqrt{2\pi}$$

12) Show that

$$\int_0^1 (x\log x)^5 dx = -\frac{5!}{6^6}$$

13) Show that

$$\int_0^\infty \frac{x^5}{5^x} dx = \frac{5!}{(\log 5)^6}$$

14) Show that

$$\int_0^{\infty} \frac{x^a}{a^x}\, dx = \frac{\overline{|(a+1)}}{(\log a)^{a+1}} \quad (a > 1)$$

15) Show that

$$\int_0^{\infty} a^{-4x^2}\, dx = \frac{\sqrt{\pi}}{4\sqrt{\log a}}$$

16) Show that

$$\int_0^{\infty} \frac{1}{3^{4x^2}}\, dx = \frac{\sqrt{\pi}}{4\sqrt{\log 3}}$$

17) Show that $\overline{|\tfrac{1}{4}}\;\overline{|\tfrac{3}{4}} = \sqrt{2}\pi,\quad \overline{|\tfrac{2}{3}}\;\overline{|\tfrac{1}{3}} = \frac{2\pi}{\sqrt{3}}$

18) Show that Prove that $\overline{|m + \tfrac{1}{2}} = \frac{(2m-1)}{2}\frac{(2m-3)}{2}\cdots\frac{1}{2}\sqrt{\pi}$ where m is a positive integer.

19) Show that

$$\int_0^1 \frac{y^{m-1} + y^{n-1}}{(1+y)^{m+n}}\, dy = \beta(m, n)$$

20) Show that $\int_0^{\infty} \frac{x^{m-1}}{(a+bx)^{m+n}}\, dx = \frac{1}{a^n b^m}\ \beta\,(m, n)$

21) Show that $\int_0^{\infty} \frac{x^8(1-x^6)}{(1+x)^{24}}\, dx = 0$

Calculus (Surface areas and Volumes of revolutions)

1) Calculate the volume of solid of revolution generated by revolving the plane area bounded by the curves $y = x^3$ and straight lines y = 0 to x = 2 about x-axis.
2) Find the surface area generated by revolving the curve $3y = x^3$ between $x = -2$ and $x = 2$ about the x-axis.
3) Derive formula for surface area of the right circular cone of height h and base radius r.
4) The region bounded by the parabola y = x^2 and the line y = 2x in the first quadrant is revolved about the y-axis to generate a solid. Evaluate the volume of the solid.
5) Calculate the volume of the solids generated by revolving the region bounded by the curve $y = 2x - 1$, $y = \sqrt{x}$ and a line $x = 0$ about y-axis.

6) A group of engineers is building a parabolic satellite dish whose shape will be formed by rotating the curve $y = ax^2$ about the y-axis. If the dish is to have a 10-ft diameter and maximum depth of 2-ft, find the value of a and the area of the surface of the dish.

7) A group of engineers is building a water tank of the shape of a paraboloid which is formed by revolving the curve $y = ax^2$ about the y-axis. If the diameter of the tank at the top is 6 meter and maximum depth of 18meter, find the value of a and the area of the surface of the tank. In your calculation include surface area of the roof of the tank.

8) A group of engineers is building a water tank of the shape of a paraboloid which is formed by revolving the curve $y = ax^2$ about the y-axis. If the diameter of the tank at the top is 6 meter and maximum depth of 18meter, find the value of a and the volume of the tank.

Double and Triple Integrals

1) Evaluate $\int_1^4 \int_0^{\sqrt{x}} \frac{3}{4} e^{y/\sqrt{x}} dy\, dx$.

2) Evaluate $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 8y dx\, dy$

3) Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dxdy}{1+x^2+y^2}$

4) Evaluate $\iint (x+y)\, dydx$ through the area enclosed by the curves $y = 2x, x - y = 2, y = 0, y = 1$.

5) Evaluate $\int_0^{\infty}\int_0^{\infty} (x^2+y^2)\, dxdy$ and hence show that $\int_0^{\infty} e^{-\pi^2} dxdy = \frac{\sqrt{\pi}}{2}$.

6) $\iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\, dxdy$ over the positive quadrant of a circle $x^2+y^2=1$.

7) $\int_0^2\int_0^2\int_0^2 (x^2+y^2+z^2)\, dzdydx$

8) $\int_{-1}^1\int_0^x\int_{x-y}^{x+y} (z-2x-y)\, dzdydx$

9) $\int_1^e\int_1^e\int_1^e \ln r \ln s \ln t\, dtdrds$

10) $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dx \, dy \, dz$

11) Evaluate $\iiint_V x^2yzdxdydz$ over the region bounded by the plane $x = 0, y = 0, z = 0, x + y + z = 1$.

Multiple Integral

1) Find the area of each of the cardioid:

   $(i)\ r = a(1 + \cos\theta)$, $(ii)\ r = a(1 - \cos\theta)$, $(iii)\ r = a(1 + \sin\theta)$,

   $(iv)\ r = a(1 - \sin\theta)$

2) Find the smaller of the areas bounded by the ellipse $x^2 + y^2 = 9$ and the line $2x + 3y = 6$.

3) Find the area of the region common to the interiors of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.

4) Find the area inside the cardioid $r = 2(1 + \cos\theta)$ and outside the circle $r = 3$.

5) Find the volume of the solid bounded by the surface $x = 0, y = 0, z = 0$ and $x + y + z = 1$.

6) Find the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = a^2$, and below by the cone $x^2 + y^2 = z^2$, (above the xy-plane).

7) Find the volume of the solid enclosed between the surface $z^2 = 2ax$ and the cylinder $x^2 + y^2 = 2ax$.

8) Find the volume of the tetrahedron bounded by the surface $x = 0, y = 0, z = 0$ and $2x + 3y + z = 6$.

9) Find the volume of the upper half of the cone $x^2 + y^2 = z^2$ cut above by the plane $z = 3$.

10) Find the Centre of mass of a thin plate of density $\rho(x, y) = x + y$ bounded by the lines $x = 0, y = x$ and the parabola $y = 2 - x^2$ in the first quadrant.

11) Find the Centre of mass of a thin plate of density $\rho(x, y) = x^2 + y$ bounded by the parabolas $y = x^2$ and $x = y^2$.

12) Find the Centre of mass of a triangular plate of density $\rho(x, y) = x^2 y$ bounded by the lines $x = 0, y = 0$ and $x + y = 1$.

Infinite Series

1. Convergence of series by Comparison Test

Ex-1: $\frac{1}{1\cdot2\cdot3} + \frac{3}{2\cdot3\cdot4} + \frac{5}{3\cdot4\cdot5} + \frac{7}{4\cdot5\cdot6} + \cdots$

Ex-2: $\sum_{n=1}^{\infty} \left[\frac{1}{n} - log\left(\frac{n+1}{n}\right)\right]$.

Ex-3: $\sum_{n=1}^{\infty} sin\frac{1}{n}$.

Ex-4: $\sum_{n=1}^{\infty} \frac{2n-1}{n(n+1)(n+2)}$.

Ex-5: $\frac{2\cdot1^3+5}{4\cdot1^5+1} + \frac{2\cdot2^3+5}{4\cdot2^5+1} + \ \dots\dots\dots + \frac{2\cdot n^3+5}{4\cdot n^5+1} + \cdots$

Ex-6: $\frac{1^2}{4^2} + \frac{5^2}{8^2} + \frac{9^2}{12^2} + \frac{13^2}{16^2} + \cdots + \frac{(4n-3)^2}{(4n)^2} + \cdots$

2. Convergence of series by De Alembert’s Ratio Test

Test the convergence by stating which test will be applicable to solve.

Ex-1: $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\frac{4}{1+2^4}+\cdots$

Ex-2: $\sum_{n=1}^{\infty}\frac{2\cdot4\cdot6\ldots(2n)}{5\cdot8\cdot11\ldots(3n+2)}$

Ex-3: $\sum_{n=1}^{\infty}\frac{2^n}{n^3+1}$.

Ex-4: $\sum_{n=1}^{\infty}\frac{n!(2)^n}{n^n}$.

Ex-5: $\sum_{n=1}^{\infty}\frac{5^n+a}{3^n+b}, a>0, b>0.$

Ex-6: $\frac{x}{1\cdot2}+\frac{x^2}{3\cdot4}+\frac{x^3}{5\cdot6}+\frac{x^4}{7\cdot8}+\cdots$

<u>3. Convergence of series by Cauchy's Root Test</u>

Test the convergence by stating which test will be applicable to solve.

Ex-1: $\sum_{n=1}^{\infty}\frac{a^{n+1}}{n}$.

Ex-2: $\frac{1^3}{3}+\frac{2^3}{3^2}+\frac{3^3}{3^3}+\frac{4^3}{3^4}+\cdots$

Ex-3: $\sum_{n=1}^{\infty}\frac{n^n x^n}{(n+1)^n}\ , x>0.$

Ex-4: $\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2x^2+\left(\frac{4}{5}\right)^3x^3+\cdots\infty.$

<u>4. Convergence of series by Cauchy's Integral Test</u>

Test the convergence by stating which test will be applicable to solve.

Ex-1: $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$.

Ex-2: $\sum_{n=1}^{\infty} ne^{-n^2}$.

Ex-3: $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$.

5. Lebnitz's Test for Alternating series

Test the convergence by stating which test will be applicable to solve.

Ex-1: $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + \cdots$

Ex-2: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2(n+1)}$.

Ex-3: $log\left(\frac{1}{2}\right) - log\left(\frac{2}{3}\right) + log\left(\frac{3}{4}\right) - log\left(\frac{4}{5}\right) + \cdots$

Ex-4: $\frac{1}{2} - \frac{2}{5} + \frac{3}{10} - \frac{4}{17} + \cdots$

1) Examine the convergence of the series
$$\frac{1}{1\cdot 2\cdot 3} + \frac{3}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \frac{7}{4\cdot 5\cdot 6} + \cdots$$
2) Test the convergence of the series
$$\frac{x}{1\cdot 2} + \frac{x^2}{3\cdot 4} + \frac{x^3}{5\cdot 6} + \frac{x^4}{7\cdot 8} + \cdots$$
3) Discuss the convergence of the series
$$\frac{1^3}{3} + \frac{2^3}{3^2} + \frac{3^3}{3^3} + \frac{4^3}{3^4} + \cdots$$
4) Check the convergence of the series
$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}.$$
5) A ball is dropped from 'a' meters above a flat surface. Each time the ball hits the surface after falling a distance 'h', it rebounds a distance 'rh' where $0 < r < 1$. Find total distance

the ball travels up and down when $a = 6$ meter and $r = \frac{2}{3}$ meter.

1. Obtain the Maclaurin's expansion of $\tan\left(\frac{\pi}{4} + x\right)$, hence find the value of $tan(46.5^0)$.

2. Expand $[\log(1 + x)]^2$ in the ascending powers of x, correct upto fourth power of x.
3. Express $(x - 2)^4 - 3(x - 2)^3 + 4(x - 2)^2 + 5$ in powers of x.
4. Express $6 - x^2 - x^3 + 11x^4$ in powers of $(x - 3)$.
5. Expand $\log \cos x$ about $\frac{\pi}{3}$ using Taylor's expansion.

1. Compute the tangent plane and normal line to $x^2 + y^2 + z^2 = 30$ at the point (1, -2, 5)..

2. Find and classify all the critical points of $f(x, y) = 4 + x^3 + y^3 - 3xy$.

Q: Locate and classify any critical points. $H(r, s) = rs + 5s^2 + r^2$

<u>Multivariable Calculus(Partial Derivatives, total derivative, Euler's theorem)</u>

a) If $u = \log r$ & $r = x^3 + y^3 - x^2y - xy^2$ then prove that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 u = -\frac{4}{(x + y)^2}$$

b) If $z^3 - xz - y = 4$, find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

c) If $v = \frac{c}{\sqrt{t}} e^{-x^2/4a^2t}$, then show that $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$

d) If $z = f(x + at) + \phi(x - at)$, then show that $\frac{\partial z}{\partial t} = a^2 \frac{\partial^2 z}{\partial x^2}$.

e) If $z^3 - xz - y = 0$, prove that $\frac{\partial^2 z}{\partial x \partial y} = -\frac{(3z^2 + x)}{(3z^2 - x)^3}$

f) Find $\frac{du}{dx}$, where $u = x \log xy$ and $x^3 + y^3 = -3xy$.

(Note: Here $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\frac{dy}{dx}$)

h) Find $\frac{dy}{dx}$ by use of partial derivatives: $x^4 + y^4 = 5a^2xy$.

i) If $u = f(r^2)$ $where$ $r^2 = x^2 + y^2 + z^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 4r^2 f''(r^2) + 6f'(r^2)$$

j) Given $u = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

k) Q: If $z^3 - xz - y = 0$, then prove that $\frac{\partial^2 z}{\partial x \partial y} = -\frac{(3z^2+x)}{(3z^2-x)^3}$

l) If u = $\sin^{-1}(x^2 + y^2)^{1/5}$, prove that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{2}{25} \tan u\,(2tan^2 u - 3).$$

m) If $Z = \tan^{-1}\left(\frac{x}{y}\right)$ & $x = 2t$, $y = 1 - t^2$ then prove that $\frac{dz}{dt} = \frac{2}{1+t^2}$

1) If $u = tan^{-1}\left[\frac{x^3+y^3}{x-y}\right]$, then prove that $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \sin 4u - \sin 2u$.

2) If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, then find the value of $xu_x + yu_y$.

3) If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then show that $xu_x + yu_y + \frac{1}{2}\cot u = 0$

4) If $u = \operatorname{cosec}^{-1}\left(\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{3}}+y^{\frac{1}{3}}}\right)^{\frac{1}{2}}$, prove that $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy} = \frac{\tan u}{144}(13 + \tan^2 u)$.

5) If $u = \tan^{-1}\left(\frac{y^2}{x}\right)$, evaluate $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}$.

First order Partial Differential Equation

1. Form the partial Differential equation by eliminating the arbitrary constants
   a) $2z = (ax+b)^2 + b$
   b) $z = xy + y\sqrt{(x-b)^2} + a$
   c) $x^2 + y^2 = (z-c)^2 \tan^2 \infty$
   d) $z = ax + (1-a)y + b$
2. Construct partial Differential equation by eliminating the arbitrary functions
   a) $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$

b) $v=\frac{1}{r}\left[f\left(r-at\right)+F\left(r+at\right)\right]$

c) $F\left(x+y+z,x^2+y^2+z^2\right)=0$

d) $F(x^2+y^2,z-xy)=0$

e) $z=F(x+iy)+F(x-iy)$

3. Solve the following equations:

a) $x(y-z)p+y(z-x)q=z(x-y)$

b) $x^2(y-z)p+y^2(z-x)q-z^2(x-y)=0$

c) $(z^2-2yz-y^2)p+(xy+zx)q=(xy-zx)$

d) $z(xp-yq)=(y^2-x^2)$

e) $z-xp-yq=a\sqrt{x^2+y^2+z^2}$

f) $p+3q=5z+\tan(y-3x)$